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Topological Masses From Broken Supersymmetry

Ignatios Antoniadis[1,†] and Tomasz R. Taylor[2]

[1] *CERN Theory Division, CH-1211 Geneva 23, Switzerland*
[2] *Department of Physics, Northeastern University, Boston, MA 02115, U.S.A.*

Abstract

We develop a formalism for computing one-loop gravitational corrections to the effective action of D-branes. In particular, we study bulk to brane mediation of supersymmetry breaking in models where supersymmetry is broken at the tree-level in the closed string sector (bulk) by Scherk-Schwarz boundary conditions, while it is realized on a collection of D-branes in a linear or non-linear way. We compute the gravitational corrections to the fermion masses $m_{1/2}$ (gauginos or goldstino) induced from the exchange of closed strings, which are non-vanishing for world-sheets with Euler characteristic -1 ("genus 3/2") due to a string diagram with one handle and one hole. We show that the corrections have a topological origin and that in general, for a small gravitino mass, the induced mass behaves as $m_{1/2} \propto g^4 m_{3/2}$, with g the gauge coupling. In generic orbifold compactifications however, this leading term vanishes as a consequence of cancellations caused by discrete symmetries, and the remainder is exponentially suppressed by a factor of $\exp(-1/\alpha' m_{3/2}^2)$.

[†] On leave from CPHT Ecole Polytechnique (UMR du CNRS 7644) F-91128, Palaiseau.

1. Introduction

There are many important reasons for studying gravitational corrections to the effective actions describing D-brane excitations. One of the phenomenologically-oriented goals presenting quite a challenging theoretical problem is to understand how supersymmetry breaking in the bulk can possibly be communicated to the gauge theory of supersymmetric branes. The world-sheet configuration that mediates such a supersymmetry breaking is a bordered Riemann surface with one hole and one handle (or two crosscaps in the presence of orientifolds) and has Euler characteristic -1: we call it a "genus 3/2" surface. In this work, we develop a formalism for such computations and study the (Majorana) fermion masses on branes generated from gravitational corrections in type II string models with supersymmetry breaking realized via Scherk-Schwarz boundary conditions [1,2,3,4,5]. The gravitino mass is then proportional to the compactification scale $1/R$, while supersymmetry remains unbroken locally on the world-volume of branes transverse to the Scherk-Schwarz (SS) direction [5].

Actually, in type I theory, there is also a pair of orientifold ($\mathcal{O}$) and anti-orientifold ($\bar{\mathcal{O}}$) planes formed, respectively, at the two endpoints of the compactification interval, requiring in general the presence of branes and anti-branes for canceling the Ramond-Ramond (RR) charge. One can then construct two types of D-brane configurations [6]: D$\mathcal{O}$ or $\bar{\text{D}}\bar{\mathcal{O}}$ preserving linearly half of the bulk supersymmetry on the brane world-volume [5], and $\bar{\text{D}}\mathcal{O}$ or D$\bar{\mathcal{O}}$ with non supersymmetric spectra but realizing non-linear supersymmetry [7,8]. Our study of gravitational corrections to Majorana fermion masses applies both to gauginos of the supersymmetric configurations, as well as to the goldstino of the non-supersymmetric models.

Since the mediation of supersymmetry breaking from the bulk is a local phenomenon, the possible existence of distinct branes and orientifolds far away in the bulk can only lead to a secondary subleading effect. Thus, for simplicity and without loss of generality, we will restrict our analysis to the oriented string case, the generalization in the presence of orientifolds being straightforward. Moreover, for the same reason, we will ignore the possible existence of distinct anti-branes which are required by RR charge conservation in the compact case. Note that the brane – antibrane system is not affected by the SS deformation. In any case, their presence can be avoided if some other direction of the transverse space is non-compact, without affecting our results.

Now consider a fermion mass term generated by the two-point function at zero momentum of two (tree-level) massless gauginos (or goldstinos) of the same (four-dimensional)

chirality. The generic oriented string diagram that contributes to this amplitude has g handles and h holes corresponding to D-brane boundaries. Obviously, one must have $h \geq 1$ in order to insert the gaugino vertices. The power of the string couplings is determined by the Euler characteristic $\chi = 2 - 2g - h$. A simple inspection of the internal $N = 2$ world-sheet superconformal charge, of which each gaugino vertex carries 3/2 units, shows that one needs at least three world-sheet supercurrent insertions to ensure charge conservation. This implies that independently of the source of bulk supersymmetry breaking, the lowest order diagram that can give non-vanishing contribution has $\chi = -1$ and "genus" $g + h/2 = 3/2$. In the oriented string case, there are two such diagrams: one with three boundaries $(g = 0, h = 3)$ and one with a handle and a boundary $(g = 1, h = 1)$. In the framework we described above, all boundaries are of the same type and thus the first diagram is supersymmetric and can be ignored. We are therefore left with the $(g = 1, h = 1)$ surface that contains the information about gravitational interactions. This world-sheet configuration will be studied in this work in great detail.

The main result of our analysis is that the gaugino mass is determined by an amplitude closely related to the topological partition function $\mathcal{F}_2$ [9,10]. Furthermore, for smooth compactifications with at most $N = 2$ supersymmetry on the branes, we obtain, in the limit of low-energy supersymmetry breaking, a fermion mass proportional to the gravitino mass $m_{3/2}$. On the other hand, within the effective field theory, the one loop gravitational correction leads obviously to two inverse powers of the Planck mass which can be canceled only if the momentum integral is quadratically divergent. Thus the leading contribution computes precisely the coefficient of this divergence, cutoff by the string scale. However, in the case of orbifolds, this term vanishes as a result of cancellations caused by discrete symmetries, and the remainder is suppressed exponentially as $\exp(-1/\alpha' m_{3/2}^2)$.

The paper is organized as follows. In Section 2, we describe the moduli space of the relevant genus 3/2 string diagram $(g = 1, h = 1)$, obtained by an appropriate involution of the genus 2 $(g = 2, h = 0)$ surface. We also obtain the left-over modular group and the corresponding fundamental domain of integration. In Section 3, we derive the partition function modified by the SS deformation and study the degeneration regions associated to the large radius limit. In Section 4, we compute the amplitude that determines the gaugino mass. We point out that it is closely related to the amplitude corresponding to the topological $\mathcal{F}_2$ term in type II theory. To simplify our discussion, we assume that the action of the SS twist is limited to the two-torus part of a $T^2 \times K_3$ compactification

manifold. Before discussing the general K_3, we consider orbifolds and find that the amplitude is zero in such compactification limits. We also derive a general expression valid for all K_3 compactifications. In Section 5, we discuss the world-sheet degeneration limit relevant to large radius compactifications and present explicit expressions for theta functions, prime-form *etc.* As a first step toward constructing more explicit and "calculable" examples, and in order to gain more insight into the general case, we reconsider orbifolds in Section 6, but now in the presence of non-vanishing VEVs for the blowing-up modes. In Section 7, we overcome another obstacle – a cancellation due to the $\mathbb{Z}_2$ symmetry of the SS mechanism, which occurs in the simplest form of SS compactifications – by considering its $\mathbb{Z}_N$ generalization à la Kounnas and Porrati [3]. This allows extracting the leading radius dependence of the gaugino mass and thus its behavior for small gravitino mass. Finally, Section 7 contains our conclusions and outlook. Some technical details are relegated to three Appendices.

2. Bordered $(g = 1, h = 1)$ Riemann Surface From the Involution of $(g = 2, h = 0)$

In this section, we study the genus 3/2 Riemann surface with one handle and one boundary, see Fig.1. The cycles $\mathbf{a}$ and $\mathbf{b}$ shown in Fig.1 are the canonical homology basis; the boundary $\mathbf{c}$ is homologous to $\mathbf{a}\mathbf{b}^{-1}\mathbf{a}^{-1}\mathbf{b}$. Such a surface can be obtained from the double torus of genus 2 by applying the world-sheet involution I that exchanges left and right movers, according to Fig.2. Its action consists of a reflection with respect to the plane of the dividing geodesics $\mathbf{c}$, which in the canonical homology basis interchanges $\mathbf{a}$ and $\mathbf{b}$ cycles, while simultaneously reverts the orientation of the later [11]: $\mathbf{a}_1 \leftrightarrow \mathbf{a}_2$ and $\mathbf{b}_1 \leftrightarrow -\mathbf{b}_2$. It follows that the involution symplectic matrix I, acting in order on blocks of $\mathbf{a}$ and $\mathbf{b}$ cycles, takes the form:

$$I \equiv \begin{pmatrix} \Gamma & 0 \\ C & -\Gamma \end{pmatrix} = \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix} , \qquad (2.1)$$

where Γ is a symmetric matrix of integers with $\Gamma^2 = 1$. In our case, it is given by the 2×2 Pauli matrix σ^1. A period matrix Ω invariant under the involution (2.1) satisfies [11]: $\bar{\Omega} = I(\Omega) = (C - \Gamma\Omega)\Gamma = -\sigma^1\Omega\sigma^1$. It can thus be put in the form

$$\Omega = \begin{pmatrix} \tau & -il \\ -il & -\bar{\tau} \end{pmatrix} , \qquad (2.2)$$

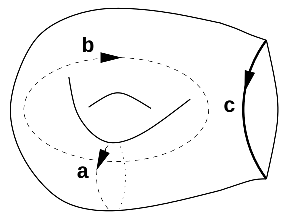

Figure 1: *Bordered $g = 1$ surface with $h = 1$ boundary.*

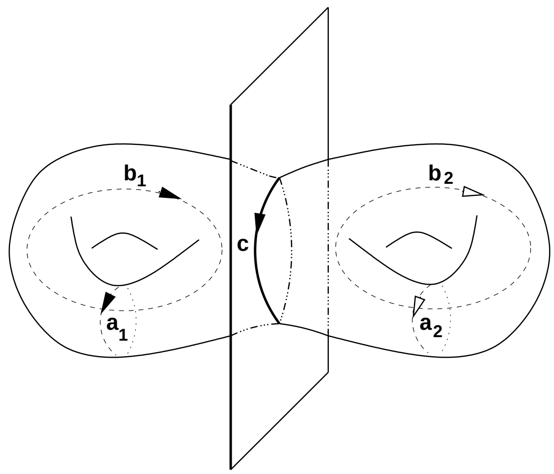

Figure 2: *The same $(g = 1, h = 1)$ surface obtained by a mirror involution of $g = 2$.*

where $\tau = \tau_1 + i\tau_2$; it depends on three real parameters τ_1, τ_2 and l.

The "relative modular group," defined by the modular transformations $\mathrm{Sp}(4, \mathbb{Z})$ that preserve the involution, was studied in general in Ref. [12]. In the case under consideration, it consists of the usual genus-one transformations that act simultaneously on the two tori of the double cover. These are generated by the familiar S and T transformations; S interchanges $\mathbf{a}$ with $\mathbf{b}$ cycles, up to a sign dictated by the involution: $\mathbf{a}_1 \leftrightarrow \mathbf{b}_1$ and $\mathbf{a}_2 \leftrightarrow -\mathbf{b}_2$; on the other hand, T leaves invariant the $\mathbf{a}$ cycles and shifts the $\mathbf{b}$ cycles by $\mathbf{b}_1 \to \mathbf{b}_1 + \mathbf{a}_1$ and $\mathbf{b}_2 \to \mathbf{b}_2 - \mathbf{a}_2$. In matrix notation, they are given by

$$S = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \qquad ; \qquad T = \begin{pmatrix} 1 & 0 \\ \sigma^3 & 1 \end{pmatrix}, \tag{2.3}$$

which, when applied to the period matrix (2.2), yield:

$$T:\ \tau \to \tau + 1\ ,\ l \to l \qquad ; \qquad S:\ \tau \to -\frac{\bar{\tau}}{|\tau|^2 - l^2}\ ,\ l \to \frac{l}{|\tau|^2 - l^2}\,, \tag{2.4}$$

or equivalently:

$$T:\ \Omega \to \Omega + \sigma^3 \qquad ; \qquad S:\ \Omega \to -\sigma^3 \Omega^{-1} \sigma^3\,. \tag{2.5}$$

Positivity of the period matrix implies that τ_2 and $-\det\Omega = |\tau|^2 - l^2$ are positive.

The fundamental domain of integration can be easily derived from (2.4). In the τ-plane, for fixed l, it consists of the part of the strip $-1/2 < \tau_1 < 1/2$ which lies above the circle of radius $\sqrt{1+l^2}$, such that $|\tau|^2 > 1 + l^2$ (see Fig.3). Obviously, the parameter l determines the size of the hole. In the limit $l \to 0$, the surface becomes a regular torus and the modular transformations (2.4) are reduced to the usual SL(2,$\mathbb{Z}$) transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$ with their familiar fundamental domain. Note that (2.4) preserve the sign of l which remains undetermined. Actually, before the involution, there is no constraint on the sign of the non-diagonal element of the period matrix Ω_{12}. However, after the involution, we show below that the sign of l is fixed to be positive [13].

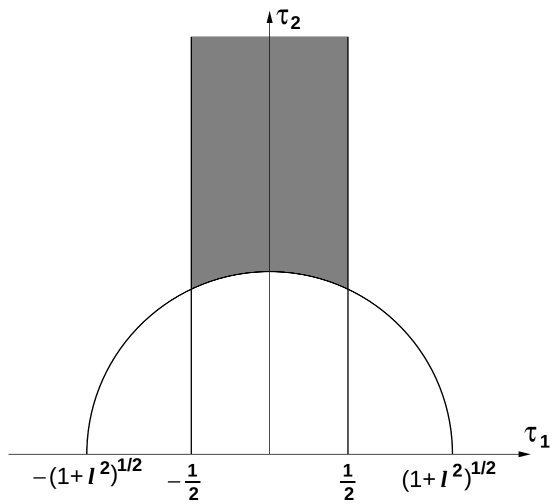

Figure 3: *The shaded region is the $(g = 1, h = 1)$ fundamental domain.*

We use the basis of two holomorphic differentials ω_i on the genus-two surface Σ, normalized to the **a** cycles:

$$\int_{\mathbf{a}_j} \omega_i = \delta_{ij} \quad ; \quad \int_{\mathbf{b}_j} \omega_i = \Omega_{ij} \quad ; \quad \int_{\Sigma} \omega_i \wedge \bar{\omega}_j = \mathrm{Im}\Omega_{ij} \, . \tag{2.6}$$

Consider now the surface Σ_1 obtained from Σ by the involution (2.1), and evaluate $\int_{\Sigma_1} \omega_i \wedge \bar{\omega}_j$ using the identity [14]:

$$\int_{\Sigma_1} \omega_i \wedge \bar{\omega}_j = \frac{i}{2} \left(\int_{\mathbf{a}_1} \omega_i \int_{\mathbf{b}_1} \bar{\omega}_j - (\mathbf{a}_1 \leftrightarrow \mathbf{b}_1) \right) + \int_{\mathbf{c}} \omega_i(z) \int^z \bar{\omega}_j \, . \tag{2.7}$$

From the involution (2.1), one deduces that on the boundary $\omega_i = \Gamma_{ij}\bar{\omega}_j$ and thus

$$\bar{\omega}_{1,2}(x) = \omega_{2,1}(x) \, , \qquad x \in \mathbf{c} \, . \tag{2.8}$$

Using this property, one has

$$\int_{\Sigma_1} \omega_i \wedge \bar{\omega}_j = \int_{\Sigma_1} \omega_i \wedge (\bar{\omega}_j - \sigma^1_{jk}\omega_k) = \frac{i}{2}\left(\int_{\mathbf{a}_1} \omega_i \int_{\mathbf{b}_1} (\bar{\omega}_j - \sigma^1_{jk}\omega_k) - (\mathbf{a}_1 \leftrightarrow \mathbf{b}_1)\right) , \quad (2.9)$$

since the boundary contribution vanishes. Using now the relations (2.6) and (2.2), we find:

$$\int_{\Sigma_1} \omega_1 \wedge \bar{\omega}_1 = \tau_2 - \frac{l}{2} \quad ; \quad \int_{\Sigma_1} \omega_2 \wedge \bar{\omega}_2 = \frac{l}{2} \quad ; \quad \int_{\Sigma_1} \omega_1 \wedge \bar{\omega}_2 = -\frac{l}{2} . \quad (2.10)$$

From the second relation above, it then follows that l is positive.

3. Partition Functions

In this section, we compute bosonic and fermionic partition functions, as well as the SS deformation, on the genus 3/2 ($g = 1, h = 1$) Riemann surface that was described above. Following the analysis of Ref. [11], the bosonic determinants can be obtained by taking the appropriate square root of the corresponding expression on the genus 2 double cover, up to a correction factor R_Σ that depends on the involution and on the boundary conditions, Neumann (N) or Dirichlet (D):[1]

$$g = 2 \ : \ (\det \mathrm{Im}\Omega)^{-1/2} |Z_1|^{-1} \quad \longrightarrow \quad (g = 1, h = 1) \ : \ (R_{\Sigma_1} \det \mathrm{Im}\Omega)^{-1/4} Z_1^{-1/2} , \quad (3.1)$$

where Z_1 is the chiral part of the determinant, and obviously after the involution the period matrix Ω has the reduced form (2.2). The correction factor is given by

$$R^{\mathrm{N}}_{\Sigma_1} = (R^{\mathrm{D}}_{\Sigma_1})^{-1} = \det\left(\frac{1-\Gamma}{2}\mathrm{Im}\Omega + \frac{1+\Gamma}{2}(\mathrm{Im}\Omega)^{-1}\right) = \frac{\tau_2 + l}{\tau_2 - l} , \quad (3.2)$$

where Γ is defined from the form of the involution (2.1) and equals σ^1 in our case. As a result, the bosonic determinant on Σ_1 becomes $(\tau_2 \pm l)^{-1/2} Z_1^{-1/2}$, where the plus (minus) sign corresponds to Neumann (Dirichlet) boundary conditions. Note that the correction factor (3.2) is invariant under the modular transformations (2.4). It will be also useful to define

$$\tilde{\tau} \equiv \tau_1 + i\sqrt{\tau_2^2 - l^2} , \quad (3.3)$$

which has the usual form of SL(2,$\mathbb{Z}$) transformations, $\tilde{\tau} \to \tilde{\tau} + 1$ and $\tilde{\tau} \to -1/\tilde{\tau}$.

[1] Note the difference in the definition of R_Σ between the two references of [11]; the one is inverse of the other. Here, we use the definition of Bianchi and Sagnotti.

The generalization to a compact boson is straightforward. The classical action (see Appendix A) is:

$$S_{\rm cl} = \frac{\pi R^2}{2}(\vec{m}+\vec{n}\bar{\Omega})({\rm Im}\Omega)^{-1}(\vec{m}^T+\Omega\vec{n}^T) = \frac{\pi R^2}{(\tau_2 \pm l)}\left(m^2+2mn\tau_1+n^2(|\tau|^2-l^2)\right) \, , \quad (3.4)$$

where R is the compactification radius in α' units, and $\vec{m} = m(1,\mp 1)$ and $\vec{n} = n(1,\pm 1)$ are the winding numbers around the **b** and **a** cycles, respectively. Their form is fixed by the involution, while the two signs correspond to N and D boundary conditions. Combining (3.4) with the quantum determinant (3.1), and summing over all classical solutions, one finds the total contribution to the partition function of a compact boson:

$$Z_B = Z_1^{-1/2}\sum_{m,n} Z^{\rm N,D}_{m,n} \quad ; \quad Z^{\rm N,D}_{m,n} = \frac{R}{(\tau_2 \pm l)^{1/2}} e^{-\pi R^2 \left(\frac{(m+n\tau_1)^2}{\tau_2 \pm l} + n^2(\tau_2 \mp l)\right)} \, . \quad (3.5)$$

After performing a Poisson resummation in m, one can bring the partition function $Z^{\rm N,D}_{m,n}$ into the Hamiltonian form $Z^{\rm N,D}_{\tilde{m},n}$:

$$Z^{\rm N,D}_{\tilde{m},n} = e^{\frac{i\pi}{2}\left(p_L^2\tilde{\tau} - p_R^2\bar{\tilde{\tau}}\right)} \quad ; \quad p_{L,R} = \frac{\tilde{m}}{\tilde{R}^{\rm N,D}} \pm n\tilde{R}^{\rm N,D} \quad {\rm with} \quad \tilde{R}^{\rm N,D} \equiv R/(R^{\rm N,D}_{\Sigma_1})^{1/4} \, , \quad (3.6)$$

where $\tilde{\tau}$ was defined in (3.3). Thus, the lattice partition function takes the familiar form in terms of an effective radius modified by the correction factor (3.2). Obviously, in the limit where the size of the hole vanishes, $l \to 0$, the correction factor becomes unity and the partition function is reduced to its toroidal form. From the expression (3.6), it is also easy to see that T-duality, $R \to 1/R$, exchanges N and D boundary conditions. Note that $R \to 1/R$ is also formally equivalent to $l \to -l$.

3.1. Fermions and Theta Functions

The method of taking the square root from the double cover can be also applied to the fermionic determinants giving rise to theta-functions. Each complex fermion leads to 16 spin structures corresponding to four boundary conditions $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ along the two non-trivial cycles **a** and **b** of Σ_1 for the left and right movers:

$$\Theta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\Omega) = \sum_{\vec{n}=(n_1,n_2)} e^{i\pi(\vec{n}+\vec{a})\Omega(\vec{n}+\vec{a})^T + 2i\pi(\vec{n}+\vec{a})\sigma^3\vec{b}^T} \, . \quad (3.7)$$

The insertion of σ^3 in the last phase was chosen for convenience, so that in the toroidal limit $l \to 0$:

$$l \to 0 : \qquad \Theta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (\Omega) \longrightarrow \Theta \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} (\tau) \, \bar{\Theta} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} (\bar{\tau}) \, . \tag{3.8}$$

It follows that under the modular transformations (2.5), Θ transforms as the product of the 1-loop theta's $\Theta\bar{\Theta}$:

$$\begin{aligned} T: \quad & \Theta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (\Omega + \sigma^3) = e^{-i\pi a_1(a_1+1)+i\pi a_2(a_2+1)} \, \Theta \begin{bmatrix} \vec{a} \\ \vec{b} + \vec{a} + \frac{1}{2} \end{bmatrix} (\Omega) \, , \\ S: \quad & \Theta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (-\sigma^3 \Omega \sigma^3) = \sqrt{\det \Omega} \; e^{2i\pi(a_1 b_1 - a_2 b_2)} \Theta \begin{bmatrix} -\vec{b} \\ \vec{a} \end{bmatrix} (\Omega) \, . \end{aligned} \tag{3.9}$$

The partition function involves a sum over spin structures with appropriate coefficients $c \left[\substack{\vec{a} \\ \vec{b}} \right]$, determined at one loop level. As usually, at higher loops the corresponding coefficients are determined by the factorization properties of the vacuum amplitude. In our case, it is sufficient to consider the $l \to 0$ limit, to deduce in a shorthand notation:

$$c \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} = c_L \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \; c_R \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \, , \tag{3.10}$$

where the subscripts L, R denote the left and right movers in the torus amplitude.

3.2. The Scherk-Schwarz Deformation and the Large Radius Limit

Here, we will consider the breaking of bulk supersymmetry via SS boundary conditions along a direction transverse to the D-brane stack [1,2,3,4,5]. In the closed string sector, it amounts to deform the partition function by coupling the lattice momenta to the fermionic spin structures, imposing antiperiodic boundary conditions to space-time fermions consistently with world-sheet modular invariance:

$$\Theta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} \longrightarrow e^{-2i\pi n(b_1 - b_2)} \; \Theta \begin{bmatrix} \vec{a} + n\vec{1} \\ \vec{b} + m\vec{1} \end{bmatrix} = \delta_s^{\rm SS} \; \Theta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} \quad ; \quad \delta_s^{\rm SS} = e^{2i\pi[m(a_1-a_2)-n(b_1-b_2)]} \, , \tag{3.11}$$

where $\vec{1} \equiv (1,1)$, and m, n are the winding numbers appearing in $Z^D_{m,n}$ of (3.5). The choice of Dirichlet conditions follows from the transversality of the SS direction to the boundary. Thus, the SS deformation consists of inserting the phase $\delta_s^{\rm SS}$ (3.11) in the partition function. In the zero winding sector, $n = 0$, upon Poisson resummation in m, it amounts to shift the Kaluza-Klein momenta $\tilde{m}$ in (3.6) of space-time fermions ($a_1 - a_2 = 1/2$) by 1/2, giving

a mass to the gravitino $1/2R$. In the odd winding n sector, on the other hand, there is a change in the GSO projection due to the additional phase.

We consider now the large radius limit $R \to \infty$, where supersymmetry is restored. The relevant partition function is $Z^D_{m,n}$, corresponding to the lower sign of (3.5). Since at the origin of the lattice, $m = n = 0$, the SS phase becomes trivial, the contribution to the gaugino mass should vanish by supersymmetry. Thus, for finite moduli (τ_2 and l) the result is exponentially suppressed in R. To avoid exponential suppression, n should vanish and τ_2 should go to infinity. This limit corresponds to the handle degeneration, describing a massless closed string exchange in the loop, and thus gravitational corrections in the effective field theory.

Note that for Neumann boundary conditions, one should get an additional contribution from all winding numbers $n \neq 0$ in the limit $\tau_2, l \to \infty$ with $\tau_2 - l \to 0$, so that $\tau_2^2 - l^2$ remains fixed. This degeneration describes precisely the non-gravitational two-loop non-planar diagram in the gauge theory of the D-branes. Obviously, this limit is not relevant for our purposes, since it does not contain any information on the mediation of supersymmetry breaking from the bulk to the boundary.

4. Gaugino Mass as a Descendant of a Topological Amplitude

In this section, we evaluate the two-point function involving two gauginos at zero momentum coupled to the boundary of a $(g = 1, h = 1)$ surface Σ_1. We consider the case with $N = 2$, $D = 4$ supersymmetry on the boundary. Thus the $D = 10$ type II theory is compactified on the six-dimensional internal space $K3 \times T^2$, where T^2 contains the Scherk-Schwarz circle transverse to the boundary direction. The $K3$ manifold is described by a (4,4) superconformal field theory, yielding $N = 4$ supersymmetry in the bulk, spontaneously broken by the SS boundary conditions. As it becomes clear in the following, the computation of the gaugino mass is closely related to the computation of the genus 2 topological amplitude $\mathcal{F}_2$ that determines the gravitational coupling W^4 in Calabi-Yau compactifications of type II theory [10]. Before a full-fledged discussion of $K3$, it is very instructive to consider its orbifold limit, with $K3$ realized as T^4/H, where $H = \mathbb{Z}_2$ reflection $(z \to -z)$ or another $\mathbb{Z}_N$ symmetry of T^4.

4.1. The Orbifold Case

The gaugino vertex operator of definite chirality α, at zero momentum, in the canonical $-1/2$ ghost picture, reads:

$$V_\alpha^{(-1/2)}(x) =: e^{-\varphi/2} S_\alpha S_{int} : , \tag{4.1}$$

where x is a position on the boundary of the world-sheet, φ is the scalar bosonizing the superghost system, and S_α (S_{int}) is the space-time (internal) spin field. Upon complexification of the four space-time fermionic coordinates, ψ_I for $I = 1, 2$, and introducing the bosonized scalars $\psi_I = e^{i\phi_I}$, one has

$$S_\alpha = e^{\pm\frac{i}{2}(\phi_1 + \phi_2)} \quad ; \quad \alpha = \pm\,. \tag{4.2}$$

Similarly, bosonizing the fermionic coordinates of T^2 and $K3$, by introducing the scalars ϕ_3 and $\phi_{4,5}$, respectively, one has:

$$S_{int} = e^{\frac{i}{2}(\phi_3 + \phi_4 + \phi_5)}\,. \tag{4.3}$$

In order to compute the amplitude involving two fermions (4.1) at the boundary of a $(g = 1, h = 1)$ surface, one has to insert three picture changing operators $e^{\varphi} T_F$, where T_F is the world-sheet supercurrent. We choose to insert all these operators on the boundary **c**. One of them is needed to change the ghost-picture of one gaugino to $+1/2$, while the other two arise from the integration over the supermoduli:

$$m_{1/2} = g_s^2 \int_{F(\Sigma_1)} d\mu(\Omega) \int_{\partial\Sigma_1} dxdy\ \mathcal{A} \quad ; \quad \mathcal{A} = \left\langle V_+^{(-1/2)}(x) V_-^{(-1/2)}(y) \prod_{a=1}^{3} e^{\varphi} T_F(z_a) \right\rangle , \tag{4.4}$$

where g_s is the string coupling. Its square includes g_s associated to Euler characteristic -1 as well as an extra g_s from the normalization of gaugino kinetic terms on the disk. The moduli integration is over the fundamental domain $F(\Sigma_1)$, shown in Fig. 3, with the appropriate measure $d\mu(\Omega)$. The dependence on the positions z_a of the picture changing operators is gauge artifact and should disappear from the physical amplitude.

By internal charge conservation, since both gauginos carry charge $+1/2$ for ϕ_3, ϕ_4 and ϕ_5, only the internal part of the world-sheet supercurrents, T_F^{int} contributes; each T_F should provide -1 charge for ϕ_3, ϕ_4 and ϕ_5, respectively. In the orbifold case, $T_F^{int} =$

$\sum_{I=3}^{5}\psi_I^*\partial X^I + c.c.$, where X^3 and $X^{4,5}$ are the complexified coordinates of T^2 and $K3$, respectively. The later transform under an element h of the orbifold group H as $X^4 \to hX^4$ and $X^5 \to h^{-1}X^5$. The amplitude (4.4) then becomes:

$$\mathcal{A} = \langle e^{-\varphi/2}(x)e^{-\varphi/2}(y)\prod_{I=3}^{5}e^{\varphi}(z_I)\rangle\prod_{I=1}^{2}\langle e^{i\phi_I/2}(x)e^{-i\phi_I/2}(y)\rangle\prod_{I=3}^{5}\langle e^{i\phi_I/2}(x)e^{i\phi_I/2}(y)e^{-i\phi_I}(z_I)\rangle, \tag{4.5}$$

where $\{z_I\}$, $I=3,4,5$, is a permutation of $\{z_a\}$, $a=1,2,3$, and an implicit summation over all permutations is understood.

Performing the contractions for a given spin structure s, one finds:

$$\begin{aligned}\mathcal{A}_s = & \frac{\theta_s^2(\frac{1}{2}(x-y))\prod_{I=3}^{5}\theta_{s,h_I}(\frac{1}{2}(x+y)-z_I)\,\partial X_{h_I}(z_I)}{\theta_s(\frac{1}{2}(x+y)-\sum_{I=3}^{5}z_I+2\Delta)}\\ & \times\frac{\sigma(x)\sigma(y)}{\prod_{I<J}^{3,4,5}E(z_I,z_J)\prod_{I=3}^{5}\sigma^2(z_I)}\times\frac{Z_2}{Z_1^4\prod_{I=3}^{5}Z_{1,h_I}}\,\delta_s^{\rm SS}\,Z_{lat}\,,\end{aligned} \tag{4.6}$$

where θ_s is the genus-two theta-function of spin structure s, E is the prime form, σ is a one-differential with no zeros or poles and Δ is the Riemann θ-constant [15,16]. Z_{1,h_I} is the (chiral) non-zero mode determinants of the h_I-twisted $(1,0)$ system, with $h_3=1$, $h_4=h$, $h_5=h^{-1}$, while $Z_1\equiv Z_{1,1}$, and Z_2 is the chiral non-zero mode determinant of the $(2,-1)$ b-c ghost system. Finally, Z_{lat} stands for all zero-mode parts of space-time and internal coordinates. In our case, it is given by:

$$Z_{lat} = \frac{1}{(\tau_2+l)^2}\,Z_{m,n}^D\,Z_{rest}\,, \tag{4.7}$$

where an implicit summation over m,n should be performed, taking into account also $\delta_s^{\rm SS}$ and the ∂X_I factors in (4.6). Recall that $\delta_s^{\rm SS}$ is the phase (3.11) of the SS deformation, $Z_{m,n}^D$ is the momentum lattice of the SS circle (3.5), satisfying Dirichlet conditions at the boundary, while Z_{rest} denotes the remaining zero-mode contribution of the $K3$ lattice, together with the additional (non-SS) direction of T^2.

The large radius limit along the SS direction was described in the previous section. In this limit, the amplitude is exponentially suppressed unless $\tau_2\to\infty$ with l fixed, corresponding to the effective field theory gravitational loop exchange. Moreover, the momentum sum is restricted to vanishing winding number n. Then as

$$R\to\infty: \quad \delta_s^{\rm SS} = (-1)^{2m\vec{a}\cdot\vec{1}} \qquad Z_{lat}\sim\frac{1}{(\tau_2+l)^2}\,Z_{m,0}^D\,Z_{rest}(\tau_2\to\infty)\,. \tag{4.8}$$

Note that for m even, the SS deformation becomes trivial hence, as shown below, the result vanishes by supersymmetry; the SS lattice sum is in fact restricted to the odd winding numbers m.

In order to perform an explicit sum over spin structures, we choose the positions of the picture-changing operators satisfying the condition [10]

$$\sum_{I=3}^{I=5} z_I = y + 2\Delta\,. \tag{4.9}$$

Then the spin structure-dependent part of the amplitudes (4.6) simplifies, yielding the following sum:

$$\mathcal{S}_{\mathcal{O}} = \sum_{\vec{a},\vec{b}} (-1)^{2m\vec{a}\cdot\vec{1}} \theta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (\tfrac{1}{2}(x-y)) \prod_{I=3}^{5} \theta_{h_I} \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (\tfrac{1}{2}(x+y) - z_I)\,, \tag{4.10}$$

where the theta-function subscripts h_I denote additional shifts in their characteristics associated to to the respective orbifold group elements. By using the Riemann identity [16], we obtain

$$\mathcal{S}_{\mathcal{O}} = \theta \begin{bmatrix} 0 \\ \frac{m}{2}\vec{1} \end{bmatrix} (x-\Delta) \prod_{I=3}^{5} \theta_{h_I^{-1}} \begin{bmatrix} 0 \\ \frac{m}{2}\vec{1} \end{bmatrix} (z_I - \Delta)\,. \tag{4.11}$$

Note that as anticipated, for m even, the above result vanishes by supersymmetry, due to the Riemann vanishing theorem, $\theta(z-\Delta) = 0\ \forall z \in \Sigma$. For m odd, the amplitude becomes

$$\begin{aligned} \mathcal{A} = \theta \begin{bmatrix} 0 \\ \frac{1}{2}\vec{1} \end{bmatrix} (x-\Delta) \frac{\sigma(x)\sigma(y)}{\prod_{I<J}^{3,4,5} E(z_I, z_J) \prod_{I=3}^{5} \sigma^2(z_I)} \times \frac{Z_2}{Z_1^4 \prod_{I=3}^{5} Z_{1,h_I}} \\ \times \prod_{I=3}^{5} \theta_{h_I^{-1}} \begin{bmatrix} 0 \\ \frac{1}{2}\vec{1} \end{bmatrix} (z_I - \Delta)\, \partial X_{h_I}(z_I) \times Z_{lat}\,. \end{aligned} \tag{4.12}$$

As mentioned before, the amplitude (4.12) contains an implicit sum over 6 permutations $\{z_I\}$ of the supercurrent insertion points $\{z_a\}$. Due to complete antisymmetry of the prefactor, this leads to the antisymmetrization of the product $\prod_{a=1}^{3} \theta_{h_I^{-1}}(z_a - \Delta)\partial X_{h_I}(z_a)$ in $\{z_a\}$. Recall that $h_3 = 1$, $h_4 = h$, $h_5 = h^{-1}$. For a $T^4/\mathbb{Z}_2$ orbifold, $h_4 = h_5$, hence the amplitude vanishes after summing over all permutations. In fact, as shown in Section 6, it also vanishes for all $T^4/\mathbb{Z}_N$ orbifolds. This cancellation is clearly due to discrete symmetries, therefore it is not expected to hold for a generic K_3 compactification.

4.2. The General K3 Case

The case of a general $K3\times T^2$ compactification can be discussed by slightly modifying most of previous computations. The underlying $N=4$ superconformal field theory (SCFT) of central charge $\hat{c}=4$ is characterized by an $SU(2)$ affine algebra at level one. This can be realized in terms of a free boson Φ compactified at the self-dual radius, that couples also to the spin structure s. The internal part (4.3) of the gaugino vertex operator (4.1) becomes:

$$S_{int}=e^{i\phi_3/2+i\Phi/\sqrt{2}}\,, \tag{4.13}$$

while the internal part of the world-sheet supercurrent reads:

$$T_F^{int}=\psi_3^*\partial X^3+e^{-i\Phi/\sqrt{2}}\hat{G}_-+c.c.\,, \tag{4.14}$$

where $\hat{G}_-$ has no singular operator product expansion with the $U(1)$ current $\partial\Phi$ and carries a conformal dimension $5/4$. The internal part of the amplitude corresponding to the last product of contractions in (4.5) then becomes:

$$\begin{aligned}\mathcal{A}_{int}=&\langle e^{i\phi_3/2}(x)e^{i\phi_3/2}(y)e^{-i\phi_3}(z_3)\rangle\langle e^{i\Phi/\sqrt{2}}(x)e^{i\Phi/\sqrt{2}}(y)e^{-i\Phi/\sqrt{2}}(z_4)e^{-i\Phi/\sqrt{2}}(z_5)\rangle\\&\times\langle\hat{G}_-(z_4)\hat{G}_-(z_5)\rangle\,,\end{aligned} \tag{4.15}$$

giving rise to the following spin structure dependent amplitude:

$$\begin{aligned}\mathcal{A}_s=&\frac{\theta_s^2(\frac{1}{2}(x-y))\theta_s(\frac{1}{2}(x+y)-z_3)Ch_{r,s}(x+y-z_4-z_5)}{\theta_s(\frac{1}{2}(x+y)-\sum_{I=3}^5 z_I+2\Delta)}\frac{E^{1/2}(z_4,z_5)\sigma(x)\sigma(y)}{\prod_{I<J}^{3,4,5}E(z_I,z_J)\prod_{I=3}^5\sigma^2(z_I)}\\&\times\frac{Z_2}{Z_1^4}\,\delta_s^{\rm SS}\,\partial X_3(z_3)\,Z_{lat\,T^2}\langle\hat{G}_-(z_4)\hat{G}_-(z_5)\rangle_r\,.\end{aligned} \tag{4.16}$$

Here, the label $r=0,1/2$ denotes the two different types of spectral flow of the $N=4$ SCFT, and a summation over r is implicit. $Z_{lat\,T^2}$ contains the zero-mode parts in the partition function of space-time and T^2 coordinates, while the $K3$ contribution is included in the last factor which is also independent of the spin structure. Finally, $Ch_{r,s}(\nu)$ is the spin structure dependent $SU(2)$ level one character:

$$\begin{aligned}Ch_{\vec{r}}\begin{bmatrix}\vec{a}\\\vec{b}\end{bmatrix}(\vec{v})=&Z_1^{-1/2}\sum_{\vec{n}}e^{2i\pi(\vec{n}+\vec{r}+\vec{a})\Omega(\vec{n}+\vec{r}+\vec{a})^T+2i\pi(\vec{n}+\vec{r}+\vec{a})\sigma^3(\vec{v}+2\vec{b})^T}\\=&Z_1^{-1/2}(-1)^{4\vec{b}\cdot(\vec{r}+\vec{a})}\theta\begin{bmatrix}\vec{r}+\vec{a}\\0\end{bmatrix}(2\Omega,\vec{v})\,.\end{aligned} \tag{4.17}$$

In order to perform an explicit sum over spin structures, we choose here the positions of the picture-changing operators satisfying the same condition as in (4.9). Then the spin structure-dependent part of the amplitude (4.16) simplifies, yielding the following sum:

$$
\begin{aligned}
\mathcal{S}_{\mathcal{K}3} = & \sum_{\vec{a},\vec{b}} (-1)^{2m\vec{a}\cdot\vec{1}} (-1)^{4\vec{b}\cdot(\vec{r}+\vec{a})} \\
& \times \theta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (\frac{1}{2}(x-y)) \theta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (\frac{1}{2}(x+y)-z_3) \theta \begin{bmatrix} \vec{r}+\vec{a} \\ 0 \end{bmatrix} (2\Omega, x+y-z_4-z_5)\,,
\end{aligned}
\tag{4.18}
$$

where we used the SS factor $\delta_s^{\rm SS} = (-1)^{2m\vec{a}\cdot\vec{1}}$ relevant to the $R \to \infty$ limit, Eq.(4.8). Now the summation over $\vec{b}$ can be performed by using the "addition" theorem [16]:

$$
\begin{aligned}
\frac{1}{2^g} \sum_{\epsilon} (-1)^{4\vec{\alpha}\cdot\vec{\epsilon}}\, \theta \begin{bmatrix} \vec{\alpha}+\vec{\beta} \\ \vec{\gamma}+\vec{\epsilon} \end{bmatrix} (u_1+u_2)\, \theta \begin{bmatrix} \vec{\alpha}+\vec{\beta} \\ \vec{\epsilon} \end{bmatrix} (u_1-u_2) & \\
= \theta \begin{bmatrix} \vec{\alpha} \\ \vec{\gamma} \end{bmatrix} (2\Omega, 2u_1)\, \theta \begin{bmatrix} \vec{\beta}+\vec{\gamma} \\ \vec{\epsilon} \end{bmatrix} (2\Omega, 2u_2)\,, &
\end{aligned}
\tag{4.19}
$$

with the result

$$
\mathcal{S}_{\mathcal{K}3} = \sum_{\vec{a}} \delta_s^{\rm SS} \theta \begin{bmatrix} \vec{r}+\vec{a} \\ 0 \end{bmatrix} (2\Omega, x{+}y{-}z_4{-}z_5)\, \theta \begin{bmatrix} \vec{r}+\vec{a} \\ 0 \end{bmatrix} (2\Omega, x{-}z_3)\, \theta \begin{bmatrix} \vec{r} \\ 0 \end{bmatrix} (2\Omega, y{-}z_3)\,. \tag{4.20}
$$

Finally, after using the inversion of (4.19) [16], we obtain

$$
\mathcal{S}_{\mathcal{K}3} = (-1)^{2m\vec{r}\cdot\vec{1}}\, \theta \begin{bmatrix} 0 \\ \frac{m}{2}\vec{1} \end{bmatrix} (x-\Delta)\, \theta \begin{bmatrix} 0 \\ \frac{m}{2}\vec{1} \end{bmatrix} (z_3-\Delta)\, \theta \begin{bmatrix} \vec{r} \\ 0 \end{bmatrix} (2\Omega, z_4+z_5-2\Delta) \tag{4.21}
$$

As expected, the above result vanishes by supersymmetry for m even, due to the Riemann vanishing theorem, $\theta(z-\Delta) = 0\ \forall z \in \Sigma$. For m odd, the amplitude becomes

$$
\begin{aligned}
\mathcal{A} = & (-1)^{2\vec{r}\cdot\vec{1}} \theta \begin{bmatrix} \vec{r} \\ 0 \end{bmatrix} (2\Omega, z_4+z_5-2\Delta) \langle \hat{G}_-(z_4)\hat{G}_-(z_5)\rangle_r E^{1/2}(z_4,z_5)\sigma(z_4)\sigma(z_5) \\
& \times \partial X_3(z_3)\widetilde{\omega}(z_3) \frac{\widetilde{\omega}(x)\sigma(y)}{\prod_{I<J}^{3,4,5} E(z_I,z_J) \prod_{I=3}^{5} \sigma^3(z_I)} \times \frac{Z_2}{Z_1^{9/2}} Z_{lat\,T^2}\,,
\end{aligned}
\tag{4.22}
$$

where

$$
\widetilde{\omega}(x) \equiv \frac{1}{2\pi i} \theta \begin{bmatrix} 0 \\ \frac{1}{2}\vec{1} \end{bmatrix} (x-\Delta)\sigma(x) \tag{4.23}
$$

is the holomorphic one-differential twisted by (-1) along the two **b**-cycles of the genus 2 double-cover, *i.e.* twisted by (-1) along the **b**-cycle of the bordered surface. In Eq.(4.22),

and in most cases below, we neglect some purely numerical (*i.e.* position- and moduli-independent) factors.

At this point, the amplitude (4.22) still seems to depend of the positions $\{z_a\}$ of the supercurrent insertions. However, it contains an implicit sum over 6 permutations $\{z_I\}$ of these points. As a result, the factor

$$B(z_I) \equiv (-1)^{2\vec{r}\cdot\vec{1}}\theta\begin{bmatrix}\vec{r}\\0\end{bmatrix}(2\Omega, z_4+z_5-2\Delta)\langle\hat{G}_-(z_4)\hat{G}_-(z_5)\rangle_r E^{1/2}(z_4,z_5)\sigma(z_4)\sigma(z_5) \\ \times\,\partial X_3(z_3)\widetilde{\omega}(z_3) \tag{4.24}$$

yields a completely antisymmetric combination $B(z_a)$ that transforms as a quadratic differential in each z_a, twisted by (-1) along the two **b**-cycles, with first order zeroes as $z_a \to z_b$. This implies that

$$B(z_a) = B\det\widetilde{h}_a(z_b)\,, \tag{4.25}$$

where B is constant (position independent) and $\widetilde{h}_a$ are the zero modes of the twisted $(2,-1)$ b-c system. For the latter, we can use the bosonisation formula [15,17]

$$\theta\begin{bmatrix}0\\\frac{1}{2}\vec{1}\end{bmatrix}(\sum_{a=1}^{a=3}z_a-3\Delta)\prod_{a<b}^{1,2,3}E(z_a,z_b)\prod_{a=1}^{a=3}\sigma^3(z_a)Z_1^{-1/2} = \det\widetilde{h}_a(z_b)\widetilde{Z}_2\,, \tag{4.26}$$

where $\widetilde{Z}_2$ is the nonzero-mode determinant of the twisted b-c system. Finally, after using all these relations, together with (4.9), we obtain:

$$\mathcal{A} = \frac{Z_2}{\widetilde{Z}_2 Z_1^5}BZ_{lat\,T^2}\,\widetilde{\omega}(x)\,\widetilde{\omega}(y)\,, \tag{4.27}$$

which does *not* depend on the supercurrent insertion points.

In order to compute the gaugino mass, the amplitude (4.27) should be integrated over the positions of the gaugino vertex operators and over the moduli of Σ_1. Since $x, y \in \mathbf{c} \sim \mathbf{ab^{-1}a^{-1}b}$,

$$\int_{\mathbf{c}}\widetilde{\omega}(x) = 2\int_{\mathbf{a}}\widetilde{\omega}(x)\,. \tag{4.28}$$

We show in the Appendix that, with the twisted differential normalized as in (4.23),

$$\lim_{\tau_2\to\infty}\int_{\mathbf{a}}\widetilde{\omega}(x) = 1\,. \tag{4.29}$$

In this way, we obtain

$$m_{1/2} = g_s^2\int d\mu(\Omega)\frac{Z_2}{\widetilde{Z}_2 Z_1^3}BZ_{lat\,T^2}\,, \tag{4.30}$$

which still remains to be integrated over the moduli of Σ_1. In the next Section and in Appendices A, B and C, we discuss the degeneration limit $\tau_2 \to \infty$, and derive explicit expressions for the integration measure and the determinants.

5. Degeneration Limit $\tau_2 \to \infty$, l fixed

In the limit $\tau_2 \to \infty$, l fixed, the two **a**-cycles of the double cover are "pinched" along the handles, see Fig.4. The limiting curve Σ_τ is a sphere with two pairs of points, (a, b) and $(\bar{a}, \bar{b})$, identified as the punctures. This curve can be constructed from a two-sphere by cutting out small disks on its surface (with diameters of order $e^{-\pi\tau_2}$), and applying the "plumbing" procedure for attaching the handles [16]. On the projective plane $\mathbb{P}^1$, the handles extend symmetrically from a to b and from $\bar{a}$ to $\bar{b}$, see Fig.5. The boundary of $\Sigma_{1\tau}$ is mapped to the real axis, $\mathrm{Im}\, z = 0$. The **a**-cycle winds around point a while the **b**-cycle runs from b to a.

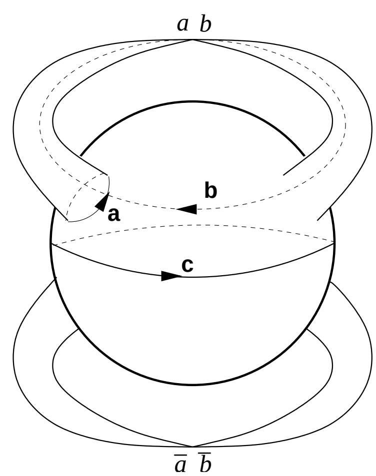

Figure 4: *The $\tau_2 \to \infty$, l fixed degeneration limit of the double-cover*

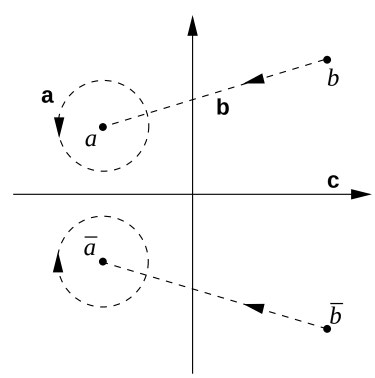

Figure 5: *Degenerate surface on the projective plane.*

On Σ_τ, the normalized basis of holomorphic one-forms is given by [16]:

$$\omega_1 = (z\,; a, b)\ , \qquad\qquad \omega_2 = (z\,; \bar{b}, \bar{a})\ , \tag{5.1}$$

where the $SL(2,C)$-covariant one-form

$$(z\,;p_1,p_2) \equiv \frac{1}{2\pi i}\left(\frac{1}{z-p_1}-\frac{1}{z-p_2}\right) . \tag{5.2}$$

Note that on the real axis, $\bar{\omega}_{1,2}=\omega_{2,1}$, as required by the involution (2.1). Furthermore, Eq.(5.1) holds up to the order $O(e^{-\pi\tau_2})$. Thus in the leading $O(1)$ order, the generalized Jacobi map becomes

$$z\in \mathbb{P}^1 \;\longrightarrow\; e^{2\pi i z_1} = \frac{z-a}{z-b}\cdot c_1(p)\ ,\ e^{2\pi i z_2} = \frac{z-\bar{b}}{z-\bar{a}}\cdot c_2(p)\ , \tag{5.3}$$

where the multiplicative constants c_1 and c_2 depend on the base point p. They will cancel inside all theta function arguments, therefore from now on we can set $c_1(p)=c_2(p)=1$. The off-diagonal period matrix element is determined by the $SL(2,C)$-invariant cross-ratio of the four puncture points:

$$e^{2\pi l} = (a,b\,;\bar{a},\bar{b}) = \frac{(a-\bar{b})(b-\bar{a})}{(a-\bar{a})(b-\bar{b})}. \tag{5.4}$$

Note that as expected, $l>0$, since a and b lie on the same half-plane of $\mathbb{P}^1$.

Naively, the $\tau_2\to\infty$ limit of the theta function seems to be trivial $[\theta(\Omega,z)\to 1]$, unless one considers an argument that diverges in this limit. Indeed, the arguments of the form $z-\Delta$ that we are interested in, have this property because $\Delta_i\propto -\Omega_{ii}/2\to\infty$. It is convenient to write

$$z-\Delta = z-\Delta_\delta+\delta(\Omega), \tag{5.5}$$

where the half-period

$$\delta(\Omega) = \frac{1}{2}\Omega\vec{1} = \frac{1}{2}\begin{pmatrix} \tau - il \\ -\bar{\tau}-il\end{pmatrix} \tag{5.6}$$

and

$$\Delta_\delta = \Delta+\delta(\Omega) \tag{5.7}$$

is the *finite* Riemann constant associated to the spin structure δ. In the leading order of the theta series expansion,

$$\theta(z+\delta) = 1+e^{-\pi l}e^{-2\pi i z_1}+e^{-\pi l}e^{-2\pi i z_2}+e^{-2\pi i(z_1+z_2)}\,, \tag{5.8}$$

with a remainder of order $O(e^{-\pi\tau_2})$. With the Jacobian map given in (5.3), we obtain

$$\theta(z-\Delta) = 1+e^{-\pi l}e^{2\pi i\Delta_{\delta_1}}\frac{z-b}{z-a}+e^{-\pi l}e^{2\pi i\Delta_{\delta_2}}\frac{z-\bar{a}}{z-\bar{b}}+e^{2\pi i(\Delta_{\delta_1}+\Delta_{\delta_2})}\frac{(z-b)(z-\bar{a})}{(z-a)(z-\bar{b})}\,. \tag{5.9}$$

Then the Riemann vanishing theorem, $\theta(z-\Delta)=0$, $\forall z \in \mathbb{P}^1$, dictates

$$e^{2\pi i\Delta_{\delta_1}} = -e^{-\pi l}\,\frac{a-\bar{b}}{b-\bar{b}}\,, \qquad e^{2\pi i\Delta_{\delta_2}} = -e^{-\pi l}\,\frac{a-\bar{b}}{a-\bar{a}}\,, \tag{5.10}$$

with l determined by Eq.(5.4).

For the prime-form, the limit can be taken in the same way as for the theta function, and yields the obvious result:

$$E(x,y) = x - y\,. \tag{5.11}$$

The differential $\sigma(z)$ is more subtle. One can start from the quotient [15]

$$\frac{\sigma(z)}{\sigma(w)} = \frac{\theta(z-p_1-p_2+\Delta)}{\theta(w-p_1-p_2+\Delta)}\,\frac{E(w,p_1)E(w,p_2)}{E(z,p_1)E(z,p_2)}\,, \tag{5.12}$$

where p_1 and p_2 are arbitrary points, and choose p_2 as the point associated to one of the odd spin structures, say $p_2 - \Delta = \delta + \binom{0}{\frac{1}{2}}$ [18]. Now it is straightforward to take a limit similar to the one that led to (5.9). It does not depend on p_1, and one finds $\sigma(z) \propto (z-b)^{-1}(z-\bar{a})^{-1}$. The constant factor is fixed by $SL(2,C)$ covariance, hence

$$\sigma(z) = (z\,;\bar{a},b)\,. \tag{5.13}$$

In Appendix B, we take into account Beltrami differentials dual to the integration measure

$$d\mu(\Omega) = d\tau_1 d\tau_2\, dl\,, \tag{5.14}$$

and we show that in the degeneration limit, the quantum determinants appearing in Eq.(4.30) satisfy:

$$\frac{Z_2}{\widetilde{Z}_2 Z_1^3} = \frac{1}{e^{2\pi l}-1}\,. \tag{5.15}$$

This leads to the gaugino mass

$$m_{1/2} = g^4 \int_{F(\Sigma_\tau)} d\tau_1 d\tau_2\, dl\, \frac{BZ_{lat\,T^2}}{e^{2\pi l}-1}\,, \tag{5.16}$$

where we replaced the string coupling by the gauge coupling g, with $g_s = g^2$.

6. Double Degeneration Limit $l \to 0$ on Blown-Up Orbifolds

It is clear from Eq.(5.16) that the dominant contribution to the gaugino mass comes from the region of small l. The $l \to 0$ limit corresponds to the "double degeneration" of the surface, with $a \to b$ on $\mathbb{P}^1$. Then the surface splits into a small neighborhood of a and b, of radius $O(l)$, which comprises the degenerate torus, and the surrounding annulus extending from the torus to the boundary; in the $l \to 0$ limit the annulus degenerates into a disk with a puncture at $z = a$. As mentioned before, l cannot become smaller than $e^{-\pi\tau_2}$, which is the cutoff provided by the diameter of the degenerate handle.

The final result for the gaugino mass (5.16) depends critically on the small l behavior of the constant B. If B vanishes as any positive power of l, then the integral over l yields a τ_2-independent constant. On the other hand, if B is constant, the integral depends logarithmically on the cutoff, bringing a factor of τ_2 which, as we will see at the end, will be converted to R^2. Recall that B was introduced in Eq.(4.25), in the context of $K3$ compactifications, and refers to a specific basis of two-differentials (C.3) given in Appendix C.

Since the analysis of a general $K3$ turns out to be quite elaborate, we defer it to another place and return to the orbifold limit. As mentioned before, the orbifold amplitude (4.12) vanishes due to symmetry with respect to the permutations of the supercurrent insertions. The simplest way to move away from the orbifold point is by inserting a vertex operator for a massless "blowing-up" mode. However, first we explain why the amplitude is zero for any $\mathbb{Z}_N$ orbifold.

6.1. Vanishing Amplitude on $\mathbb{Z}_N$ orbifolds

After using the explicit parameterization of the bosonic zero modes, Eqs.(A.1) and (A.5), and summing over all permutations of the supercurrent insertion points and over all twisted sectors,[2] one finds that the amplitude (4.12) contains the factor

$$B_O(z_a) = \sum_{k=1}^{N-1} \sum_{(a_k,b_k)} \sum_{i=1,2} L_k L_{N-k} L_i \, B_{k,i}(z_a) \,, \tag{6.1}$$

where k labels the group element $e^{2\pi i k/N}$, (a_k, b_k) labels the associated twists along the **a**- and **b**-cycles, and

$$B_{k,i}(z_a) = \det \left[\omega_i \, \widetilde{\omega}(z_1) \,, \, \omega_k \widetilde{\omega}_{N-k}(z_2) \,, \, \omega_{N-k} \widetilde{\omega}_k(z_3) \right] . \tag{6.2}$$

[2] The contribution of the untwisted sector drops out after symmetrization.

First, consider twists that are non-trivial only along the **b**-cycle, as discussed explicitly in Appendix A. Then the determinant (6.2) is exactly the same as the determinant computed in Appendix C. For $\omega_i = \omega_D$ it is zero, while for $\omega_i = \omega_N$ it is the determinant (C.4), with a k-dependent prefactor $\mathrm{Im}(h) = \sin(2\pi ik/N)$. Although Neumann boundary conditions in the SS direction are beyond the scope of the present discussion, it is worth mentioning that, even in that case, the final result would vanish, because the contribution of the kth group element in (6.1) would be canceled by the $(N-k)$th. For N even, the remaining contribution of $k = N/2$ vanishes by the same argument as for the $\mathbb{Z}_2$ orbifold mentioned in Section 4. Next, consider twists that are non-trivial only along the **a**-cycle. The corresponding twisted differentials are easy to construct: they have the property that $\omega_k \widetilde{\omega}_{N-k} \propto \omega_{N-k}\widetilde{\omega}_k$, therefore the determinant (6.2) vanishes also in this case. Similar property holds for twists that are non-trivial along both **a**- and **b**-cycles. Hence

$$B_O(z_a) = 0\,, \tag{6.3}$$

where the above equation holds to the same $O(e^{-\pi\tau_2})$ accuracy as the expansions of the holomorphic differentials.

6.2. Blowing-up Mode Insertion

We now insert in the amplitude a blowing-up mode B at zero momentum, associated to the twisted sector (h, h^{-1}) with $h = e^{2i\pi\epsilon}$ and $\epsilon = k/N$. Its vertex in the -1 ghost picture is:

$$V_B^{(-1,-1)}(\zeta,\bar\zeta) =: e^{i\epsilon(\phi_4-\bar\phi_4)+i(1-\epsilon)(\phi_5-\bar\phi_5)}\sigma_4^{--}\sigma_5^{--} :\,, \tag{6.4}$$

where σ_I^{--} is the corresponding twist field of conformal dimension $\epsilon(1-\epsilon)/2$ in both left and right movers. Using the $N=2$ world-sheet supercurrent, one finds the blowing-up vertex operator in the 0-ghost picture:

$$\begin{aligned} V_B^{(0,0)}(\zeta,\bar\zeta) = :\, & e^{-i(1-\epsilon)(\phi_4-\phi_5-\bar\phi_4+\bar\phi_5)}\sigma_4^{++}\sigma_5^{--} + e^{-i(1-\epsilon)(\phi_4-\phi_5)-i\epsilon(\bar\phi_4-\bar\phi_5)}\sigma_4^{+-}\sigma_5^{-+} \\ & + e^{i\epsilon(\phi_4-\phi_5)+i(1-\epsilon)(\bar\phi_4-\bar\phi_5)}\sigma_4^{-+}\sigma_5^{+-} + e^{i\epsilon(\phi_4-\phi_5-\bar\phi_4+\bar\phi_5)}\sigma_4^{--}\sigma_5^{++} :\,, \end{aligned} \tag{6.5}$$

where we used the short distance (OPE) expansions [19]: $\sigma_I^{--}(z,\bar z)\partial X_I(w) \sim (z-w)^{-1+\epsilon}\sigma_I^{+-}$ and $\sigma_I^{--}(z,\bar z)\bar\partial X_I^*(w) \sim (\bar z - \bar w)^{-1+\epsilon}\sigma_I^{-+}$.[3] Going from the $-$ to the $+$ component of the twist field, its conformal dimension is increased by the corresponding twist (ϵ or $1-\epsilon$).

[3] The left-right symmetric twist field σ^{++} is called τ in Ref. [19].

After inserting the blowing-up vertex $\int d^2\zeta V_B^{(0,0)}(\zeta,\bar\zeta)$, the $K3$ part of the amplitude, corresponding to the last factor of Eq. (4.5), becomes:

$$
\begin{aligned}
&\left\langle e^{i\phi_4/2}(x)e^{i\phi_4/2}(y)e^{-i\phi_4}(z_4)e^{-i(1-\epsilon)\phi_4}(\zeta)e^{i(1-\epsilon)\bar\phi_4}(\bar\zeta)\right\rangle_s \left\langle\partial X^4(z_4)\sigma_4^{++}(\zeta,\bar\zeta)\right\rangle \times\\
&\left\langle e^{i\phi_5/2}(x)e^{i\phi_5/2}(y)e^{-i\phi_5}(z_5)e^{i(1-\epsilon)\phi_5}(\zeta)e^{-i(1-\epsilon)\bar\phi_5}(\bar\zeta)\right\rangle_s \left\langle\partial X^5(z_5)\sigma_5^{--}(\zeta,\bar\zeta)\right\rangle\\
+&\left\langle e^{i\phi_4/2}(x)e^{i\phi_4/2}(y)e^{-i\phi_4}(z_4)e^{i\epsilon\phi_4}(\zeta)e^{-i\epsilon\bar\phi_4}(\bar\zeta)\right\rangle_s \left\langle\partial X^4(z_4)\sigma_4^{--}(\zeta,\bar\zeta)\right\rangle \times\\
&\left\langle e^{i\phi_5/2}(x)e^{i\phi_5/2}(y)e^{-i\phi_5}(z_5)e^{-i\epsilon\phi_5}(\zeta)e^{i\epsilon\bar\phi_5}(\bar\zeta)\right\rangle_s \left\langle\partial X^5(z_5)\sigma_5^{++}(\zeta,\bar\zeta)\right\rangle\\
\sim\ &\theta_{s,h_4}\left(\frac12(x+y)-z_4-(1-\epsilon)(\zeta-\bar\zeta)\right)\theta_{s,h_5}\left(\frac12(x+y)-z_5+(1-\epsilon)(\zeta-\bar\zeta)\right)\times\\
&\left[\frac{E(z_4,\zeta)E(z_5,\bar\zeta)}{E(z_4,\bar\zeta)E(z_5,\zeta)}\right]^{1-\epsilon}\frac{1}{E(\zeta,\bar\zeta)^{2(1-\epsilon)^2}}\left\langle\partial X^4(z_4)\sigma_4^{++}(\zeta,\bar\zeta)\right\rangle\left\langle\partial X^5(z_5)\sigma_5^{--}(\zeta,\bar\zeta)\right\rangle\\
+\ &\theta_{s,h_4}\left(\frac12(x+y)-z_4+\epsilon(\zeta-\bar\zeta)\right)\theta_{s,h_5}\left(\frac12(x+y)-z_5-\epsilon(\zeta-\bar\zeta)\right)\times\\
&\left[\frac{E(z_4,\bar\zeta)E(z_5,\zeta)}{E(z_4,\zeta)E(z_5,\bar\zeta)}\right]^{\epsilon}\frac{1}{E(\zeta,\bar\zeta)^{2\epsilon^2}}\left\langle\partial X^4(z_4)\sigma_4^{--}(\zeta,\bar\zeta)\right\rangle\left\langle\partial X^5(z_5)\sigma_5^{++}(\zeta,\bar\zeta)\right\rangle\ ,
\end{aligned}
\tag{6.6}
$$

where the expression after the $\sim$ symbol replaces the product of the last two theta-functions in Eq. (4.6), together with the $\prod_{I=4,5}\partial X^I$ factors from the two supercurrent insertions.

The spin structure sum can be performed using the same gauge condition (4.9) with the result:

$$
\begin{aligned}
&\theta_{h_4^{-1}}\begin{bmatrix}0\\ \frac{m}{2}\vec 1\end{bmatrix}\left(z_4+\epsilon(\zeta-\bar\zeta)-\Delta\right)\theta_{h_5^{-1}}\begin{bmatrix}0\\ \frac{m}{2}\vec 1\end{bmatrix}\left(z_5-\epsilon(\zeta-\bar\zeta)-\Delta\right)\\
&\left[\frac{E(z_4,\bar\zeta)E(z_5,\zeta)}{E(z_4,\zeta)E(z_5,\bar\zeta)}\right]^{\epsilon}\frac{1}{E(\zeta,\bar\zeta)^{2\epsilon^2}}\left\langle\partial X^4(z_4)\sigma_4^{--}(\zeta,\bar\zeta)\right\rangle_{h_4}\left\langle\partial X^5(z_5)\sigma_5^{++}(\zeta,\bar\zeta)\right\rangle_{h_5}\\
&+(4\leftrightarrow 5,\epsilon\leftrightarrow 1-\epsilon)\,,
\end{aligned}
\tag{6.7}
$$

which has to replace the last two terms in the product of Eq. (4.12).

We want to show that after integrating Eq.(6.7) over the vertex position ζ, the result is *not* symmetric under the transposition $z_4\leftrightarrow z_5$; this will be true even in the untwisted sector with $h_4=h_5=1$. To that end, we can go directly to the double degeneration limit $l\to 0$. Recall that the vertex position is to be integrated over the whole half-plane, $\mathrm{Im}\zeta>0$. If one is interested only in terms that are non-vanishing in the $l\to 0$ limit, one can neglect the small [radius $O(l)$] neighborhood of the puncture points. This means that the contribution of the factorized torus is negligible and that the vertex is *de facto* inserted on a hemi-sphere with one puncture point a. Then one can set $\zeta=0$ in the arguments

of theta functions. Furthermore, the twist field correlators should be evaluated on such a hemisphere. Due to the $\mathbb{Z}_N$ symmetry of the lattice, only the "quantum" parts of these correlators [20] contribute to the amplitude.[4] Their form is completely determined by OPE and $SL(2,R)$ invariance [19,20]. Without going into much detail, it is clear that Eq.(6.7) is not symmetric in z_4, z_5: in the last line, $z_4 \leftrightarrow z_5$ is accompanied by $\epsilon \leftrightarrow 1-\epsilon$, and this asymmetry should survive the ζ-integration. With the twist correlators non-vanishing on a punctured hemisphere, the constant $B(l=0)$ will be non-zero on blown-up orbifolds.

7. Radius Dependence of the Gaugino Mass

We return to the gaugino mass (5.16), now turning our attention to the dependence of the amplitude on the string configuration on T^2. Eq.(4.24) contains the respective zero mode, $\partial X_3 = \partial X + i\partial X'$, where X is the coordinate in the SS direction and X' is the "non-SS" coordinate on T^2. For the zero-mode configurations parameterized as in Eq.(A.1),

$$\partial X(z) = 2\pi R \sum_{i=1,2} L_i\, \omega_i(z)\,, \tag{7.1}$$

where the constants L_i depend on the winding numbers m and n, see Eq.(A.3). The SS coordinate satisfies D boundary conditions and, as explained before, in the $\tau_2 \to \infty$ limit all $n \neq 0$ modes are exponentially suppressed by the partition function, hence

$$\partial X = \frac{m\pi R}{i(\tau_2 - l)}\, \omega_D(z)\ , \qquad \omega_D(z) = \omega_1(z) + \omega_2(z). \tag{7.2}$$

Furthermore, non-trivial $\partial X'$ modes are also exponentially suppressed in this limit, unless the corresponding radius blows up in the same rate as R. Since the final result is the same, we will not consider here such a "fine-tuned" scenario. In this way, we obtain

$$B \propto m \frac{R}{\tau_2 - l} \tag{7.3}$$

where we ignored the factors associated to $K3$ which had already been discussed in the previous section.

---

[4] Actually, the vanishing due to the $\mathbb{Z}_N$ symmetry can be avoided by turning on a non-diagonal modulus mixing the two planes. However, the resulting contribution would be exponentially suppressed in the $\tau_2 \to \infty$ limit.

The zero modes are weighted by the torus partition function. After combining Eq.(7.3) with (4.8) and (3.5), we obtain

$$B Z_{lat\, T^2} \propto \frac{R^2}{(\tau_2 - l)^{3/2}} \frac{m}{(\tau_2 + l)^2} \, \exp(-\frac{m^2 \pi R^2}{\tau_2 - l}) \, . \tag{7.4}$$

It is clear that the amplitude vanishes after summing over the windings, due to the $\mathbb{Z}_2$ symmetry $m \to -m$. Since this symmetry is absent in the presence of Wilson lines, one could try to insert into the amplitude (4.27) a zero-momentum vertex for the Wilson line modulus ϕ:

$$\begin{aligned} \widetilde{\omega}(x)\, \widetilde{\omega}(y) \longrightarrow & \widetilde{\omega}(x) \big[\phi \int_y^x du \, \partial X_3(u) \big] \, \widetilde{\omega}(y) \\ \simeq & \widetilde{\omega}(x) \widetilde{\omega}(y) \int_y^x du \, \omega_D(u) \\ \simeq & \widetilde{\omega}(x) \widetilde{\omega}(y) \left[\Omega_D(x) - \Omega_D(y) \right] \, , \end{aligned} \tag{7.5}$$

where in the second line we replaced $\partial X_3(u)$ by its zero mode (7.2), since it cannot be contracted with other operators, while in the third line we used the property that $\omega_D(u)$ is a closed one-form to write $\omega_D(u) \equiv d\Omega_D(u)$. The above result obviously vanishes again upon integration over x and y.

7.1. Generalized Scherk-Schwarz Deformation

To avoid the vanishing of the summation over windings of the SS circle, we consider a $\mathbb{Z}_N$ SS deformation which is a straightforward generalization of the $\mathbb{Z}_2$ case [3]. For instance, consider compactifications on the product space of a $K3$ orbifold times T^2, and define the internal rotation current J, in one of the two complex coordinates of $K3$:

$$J = \psi_4^* \psi_4 + X_4^* \overleftrightarrow{\partial} X_4 \, . \tag{7.6}$$

Obviously, only discrete rotations remain that correspond to symmetries of the compactification lattice, generated by $Q = \exp\{2i\pi e \oint J\}$ with $e = 1/N$ for a $\mathbb{Z}_N$ symmetry. A generalized SS deformation can then me obtained by coupling the rotation charges with the winding numbers of a circle from T^2 (or with the whole T^2 momentum lattice in general). The corresponding partition function is obtained by replacing:

$$\frac{\Theta\left[{\vec{a} \atop \vec{b}}\right]}{Z_1\left[{h_{\mathbf{a}} \atop h_{\mathbf{b}}}\right]} \longrightarrow e^{-2i\pi e n (b_1 - b_2)} \, \frac{\Theta\left[{\vec{a} + n e \vec{1} \atop \vec{b} + m e \vec{1}}\right]}{Z_1\left[{h_{\mathbf{a}} + n e \vec{1} \atop h_{\mathbf{b}} + m e \vec{1}}\right]} \, , \tag{7.7}$$

where Z_1 denotes the contribution of the $(h_{\mathbf{a}}, h_{\mathbf{b}})$ twisted sector. Eq. (7.7) generalizes (3.11), which is recovered by setting $e = 1$, corresponding to a 2π rotation. In this case, the bosonic part is not modified and the deformation amounts to inserting the phase $\delta_s^{\rm SS}$.

Following the reasoning of Section 3, in the large radius limit, only vanishing windings $n = 0$ may give corrections that escape exponential suppression in the degeneration limit $\tau_2 \to \infty$. Thus, only the boundary conditions along the $\mathbf{b}$ cycle are modified in (7.7), and the spin structure-dependent part of the amplitude (4.10) (with the picture-changing insertion points satisfying the same gauge condition (4.9)) becomes:

$$\begin{aligned}\mathcal{S}_{\mathcal{O}} \longrightarrow \sum_{\vec{a},\vec{b}} \quad & \theta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\tfrac{1}{2}(x-y))\,\theta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\tfrac{1}{2}(x+y)-z_3)\\ & \times \theta_h\begin{bmatrix}\vec{a}\\ \vec{b}+me\vec{1}\end{bmatrix}(\tfrac{1}{2}(x+y)-z_4)\,\theta_{h^{-1}}\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\tfrac{1}{2}(x+y)-z_5)\,,\end{aligned} \tag{7.8}$$

where we defined, as in section 4.1, $h_4 \equiv h$ and thus $h_3 = 1$ and $h_5 = h^{-1}$ for an internal space $T^2 \times T^4/H$. The spin structure sum can be performed using the Riemann identity with the result:

$$\mathcal{S}_{\mathcal{O}} \longrightarrow \theta\begin{bmatrix}0\\ \frac{m}{2}e\vec{1}\end{bmatrix}(x-\Delta)\,\theta\begin{bmatrix}0\\ \frac{m}{2}e\vec{1}\end{bmatrix}(z_3-\Delta)\,\theta_{h^{-1}}\begin{bmatrix}0\\ -\frac{m}{2}e\vec{1}\end{bmatrix}(z_4-\Delta)\,\theta_h\begin{bmatrix}0\\ \frac{m}{2}e\vec{1}\end{bmatrix}(z_5-\Delta)\,. \tag{7.9}$$

After incorporating the remaining factors, one finds that the amplitude (4.12) becomes:

$$\begin{aligned}\mathcal{A} \longrightarrow\; & \theta\begin{bmatrix}0\\ \frac{m}{2}e\vec{1}\end{bmatrix}(x-\Delta)\frac{\sigma(x)\sigma(y)}{\prod_{I<J}^{3,4,5}E(z_I,z_J)\prod_{I=3}^{5}\sigma^2(z_I)} \times \frac{Z_2}{Z_1^5 Z_{1,h\oplus me}Z_{1,h^{-1}}}\\ & \times\theta\begin{bmatrix}0\\ \frac{m}{2}e\vec{1}\end{bmatrix}(z_3-\Delta)\theta_{h^{-1}}\begin{bmatrix}0\\ -\frac{m}{2}e\vec{1}\end{bmatrix}(z_4-\Delta)\,\theta_h\begin{bmatrix}0\\ \frac{m}{2}e\vec{1}\end{bmatrix}(z_5-\Delta)\\ & \times\partial X^3(z_3)\,\partial X^4_{h\oplus me}(z_4)\,\partial X^5_{h^{-1}}(z_5)\times Z_{lat}\,,\end{aligned} \tag{7.10}$$

where the symbol $h \oplus me$ denotes the addition of me to the twist $h \equiv (h_{\mathbf{a}}, h_{\mathbf{b}})$ along the $\mathbf{b}$ cycle only: $h \oplus me \equiv (h_{\mathbf{a}}, h_{\mathbf{b}}e^{2i\pi me})$. Note that the $\mathbf{b}$-cycle periodicity conditions for all z_I's, *including* z_4, and x are twisted by the same $\mathbb{Z}_{2N}$ group element $e^{i\pi ke} = e^{i\pi k/N}$ for $m = k\ mod\ 2N$. The contribution of the $k = 0$ class of windings disappears by Riemann vanishing theorem (supersymmetry), while the class of $k = N$ has the same structure as odd windings in the standard SS mechanism, and disappears after summation as a consequence of $m \to -m$ symmetry. Since the zero modes of ∂X^3 are linear in these winding numbers, the remaining contributions of $m = k\ mod\ 2N$ come with opposite sign

to those of $m = 2N - k \ mod \ 2N$; however, as explained below, they are weighted by different factors.

Starting from the amplitude (7.10) and repeating the same steps that led us in Section 4 to the final expression (4.27): antisymmetrization, bosonisation formula, *etc.*, for each class of $k \ mod \ 2N$ windings, one obtains a similar expression, but now with the tilde representing a twist by $e^{i\pi k/N}$ along the $\mathbf{b}$ cycle, *i.e.*

$$\widetilde{\omega}(x) \longrightarrow \omega_k(x) \ , \qquad \widetilde{\omega}(y) \longrightarrow \omega_k(y) \, . \tag{7.11}$$

The integration over the vertex positions yields

$$\left(\int_{\mathbf{c}} \omega_k(x) \right)^2 = (1 - e^{i\pi k/N})^2 . \tag{7.12}$$

Note that the class of $2N - k \ mod \ 2N$ windings (which come with the opposite sign) is weighted by the complex conjugate of the above factor. Now it is clear that the sum over the SS winding numbers does not vanish. However, one still needs to blow up singularities as in Section 6, in order to avoid cancellations due to similar symmetries of the $K3$ orbifold sector.

7.2. Determination of the Gaugino Mass

At this point, we can exclude any "accidental" cancellation and return to Eq.(7.4), assuming that the sum over m is restricted to one of the $m = k \ mod \ 2N$ $(k = 1, \ldots, N-1)$ class of windings. Then Eq.(5.16) gives

$$m_{1/2} \propto g^4 \int d\tau_2 \int \frac{dl}{(e^{2\pi l} - 1)} \frac{1}{(\tau_2 - l)^{3/2}} \sum_m \frac{mR^2}{(\tau_2 + l)^2} \exp(-\frac{m^2 \pi R^2}{\tau_2 - l}) \, . \tag{7.13}$$

The integral over l has a logarithmic divergence at $l = 0$ which, as explained in Section 6, is regulated by $e^{-\pi\tau_2}$. This brings a (large) factor of τ_2, while l can be set zero everywhere else in the integrand, so that

$$m_{1/2} \propto g^4 \int \frac{d\tau_2}{\tau_2^{5/2}} \sum_m mR^2 \exp(-\frac{m^2 \pi R^2}{\tau_2}) \, . \tag{7.14}$$

After rescaling $\tau_2 \to \tau_2 R^2$, the above expression yields

$$m_{1/2} \ \propto \ \frac{g^4}{R} \ \propto \ g^4 m_{3/2} \, . \tag{7.15}$$

The mass (7.15) can be understood within the effective field theory by looking at a generic one-loop graph involving a gravitational exchange. Each vertex brings one power of the Plank mass in the denominator, and thus $m_{1/2} \simeq m_{3/2}^3/M_P^2$ if the loop momentum integral is convergent [21]. However if, as naïvely expected, the momentum integral is quadratically divergent, then one obtains $m_{1/2} \simeq m_{3/2}\,\Lambda_{UV}^2/M_P^2$. Since the ultraviolet cutoff $\Lambda_{UV} \propto M_P$, the result (7.15) confirms this expectation. Note, however, that in the case of orbifold compactifications, for which $m_{1/2} = \mathcal{O}(e^{-1/\alpha' m_{3/2}^2})$, such a crude argument fails because it ignores the effects of discrete symmetries.

8. Conclusions and Outlook

In this work, we developed a formalism for computing one-loop gravitational corrections to the effective action of D-branes. Furthermore, we have shown that the genus 3/2 amplitude responsible for communicating the bulk supersymmetry breaking to open string fermions is closely related to the well-known genus 2 topological term $\mathcal{F}_2$. For models with large extra dimensions and low-energy supersymmetry breaking à la Scherk-Schwarz, this mass is proportional to the gravitino mass. However, the result is zero in the case of the standard, $\mathbb{Z}_2$-symmetric, Scherk-Schwarz mechanism, hence one is forced to consider its $\mathbb{Z}_N$ generalization in order to generate a non-vanishing gaugino mass.

In fact, it is not easy to find explicit examples of such a mass generation in "calculable" orbifold models. Orbifolds have discrete symmetries that generically lead to further cancellations. These, in turn, can be avoided by blowing up the orbifold singularities, which can be accomplished by switching on non-zero VEVs of certain twisted fields. This mass generation mechanism clearly needs a deeper explanation. On one hand, one should find a more profound connection to the underlying world-sheet theory. On the other hand, it would be very useful to have a more precise field-theoretical description in terms of effective interactions and Feynman diagrams. Both the topological origin as well as the relation to quadratic divergences strongly suggest that the mass is generated by anomalies, but more work is needed to identify a direct link.

One could think that the gaugino masses are due to the so-called "anomaly-mediation" [22] of the four-dimensional (super)conformal anomaly. However, such a mechanism predicts one-loop masses proportional to g^2 and not to g^4 as we find. Furthermore, the anomaly-mediated gaugino masses are proportional to the beta functions, therefore they would require the presence of at least one more world-sheet boundary. Thus the mismatch

of the powers of gauge coupling and of the group-theoretical coefficients hints against superconformal anomaly mediation. In fact, our results raise an interesting question whether the anomaly mediation mechanism can be implemented at all in any theory with a sensible ultraviolet completion.

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Appendix A. Bosonic Zero Modes and Twisted Differentials

The amplitude that determines the gaugino mass includes three supercurrent insertions, with one of them effectively coupled to the the SS torus T^2 and the other two to the $K3$ surface. The T^2 contribution involves an explicit factor of the (untwisted) zero mode ∂X_3; in the case of orbifold compactifications, the other two insertions involve also the zero modes associated to the twisted sectors, see respectively Eqs.(4.16) and (4.6).

In the case of untwisted direction, it is convenient to consider separately the two real components, each of the form

$$X(P) = 2\pi R \sum_{i=1,2} [L_i \int_O^P \omega_i(z) + \bar{L}_i \int_O^P \bar{\omega}_i(\bar{z})]\,, \tag{A.1}$$

where O is an arbitrary base point. Due to the reality property (2.8), the Neumann and Dirichlet boundary conditions read, respectively,

$$\bar{L}_{1,2} = \pm L_{2,1} \qquad \begin{cases} + \text{ for } N \\ - \text{ for } D \end{cases} \tag{A.2}$$

By considering the boundary conditions for a string winding n times around the $\mathbf{a}$-cycle and m times around the $\mathbf{b}$-cycle, we obtain

$$L_1 = \frac{m + n[\tau_1 - i(\tau_2 \pm l)]}{2i(\tau_2 \pm l)}\,, \qquad L_2 = \pm \bar{L}_1\,. \tag{A.3}$$

The classical action, $S_{\rm cl} = \frac{1}{2\pi\alpha'}\int_{\Sigma_1} \partial_z X \partial_{\bar{z}} X$, can be computed by using Eqs.(2.10), with the result written in Eq.(3.4). As explained in Section 3, in the degeneration limit $\tau_2 \to \infty$, l fixed, the $n \neq 0$ modes are exponentially suppressed by the partition function, therefore the relevant zero modes are given by

$$\partial_z X_{N,D} = \frac{m\pi R}{i(\tau_2 \pm l)}\omega_{N,D}\,, \qquad \omega_{N,D} = \omega_1(z) \mp \omega_2(z)\,. \tag{A.4}$$

In order to construct the zero modes twisted by a $\mathbb{Z}_N$ group element h, we can start from the $g = 2$ double cover Σ, with the twisted *complex* instanton configuration [23]

$$X_h(P) = 2\pi R[L_h \int_O^P \omega_h(z) + \bar{L}_h \int_O^P \bar{\omega}_{h^*}(\bar{z})]\,, \tag{A.5}$$

where $\omega_h(z)$ is the (unique) holomorphic one-differential twisted by (h_{a_i}, h_{b_i}), $i = 1, 2$, along the $\mathbf{a}$- and $\mathbf{b}$-cycles of Σ, respectively. The constants L_h and $\bar{L}_h$ are determined by the boundary conditions. Since we are interested in the instanton solutions on Σ_1, we identify $h_{a_1} = h_{\mathbf{a}}$, $h_{b_1} = h_{\mathbf{b}}$. For our purposes, it is completely sufficient to consider configurations with a non-trivial twist only along the $\mathbf{b}$-cycle: $h_{\mathbf{a}} = 1$, $h_{\mathbf{b}} \equiv h$.

The simplest way to obtain the $\tau_2 \to \infty$ limits of twisted differentials is by working directly in the degeneration limit of the punctured hemisphere. The (unique) $SL(2, C)$-covariant, holomorphic differential twisted by h along the $\mathbf{b}$-cycle is given by:

$$\omega_h(z) = (z; a, \bar{a}) - h\,(z; b, \bar{b})\,, \tag{A.6}$$

where $(z; p_1, p_2)$ is defined in (5.1). If an additional twist by (-1) along the $\mathbf{b}$-cycle is present, then

$$\widetilde{\omega}_h(z) = (z; a, \bar{a}) + h\,(z; b, \bar{b})\,. \tag{A.7}$$

It is easy to see that $\widetilde{\omega}(z) = \widetilde{\omega}_{h=1}(z)$ given by the above formula does indeed appear in the $\tau_2 \to \infty$ limit of Eq.(4.23). This also proves Eq.(4.29). Note that at the boundary,

$$\omega_h(x) = \bar{\omega}_{h^*}(x)\,, \qquad x \in \partial\Sigma_1, \tag{A.8}$$

therefore the Neumann and Dirichlet boundary conditions read, respectively,

$$\bar{L}_h = \pm L_h \qquad \begin{cases} + \text{ for } N \\ - \text{ for } D \end{cases}\,. \tag{A.9}$$

Since the amplitude considered in the paper vanishes for orbifold compactifications, we do not pursue the discussion of twisted instanton configurations any further. We will make use, however, of the twisted differentials in the following computation of $\widetilde{Z}_2$.

Appendix B. Moduli Integration Measure

The integration measure over the moduli of the $(g=1, h=1)$ Riemann surface under consideration is induced by the chiral measure of its $g=2$ double-cover. It involves 3 insertions of the reparametrization ghosts b necessary to soak up the corresponding zero modes:

$$d\mu(\Omega) = d\tau d\bar{\tau} dl \; Z_2^{-1} \epsilon^{a_1 a_2 a_3} \int \left\langle \prod_{p=1}^{p=3} dw_p \, \mu_{a_p}(w_p) b(w_p) \right\rangle , \tag{B.1}$$

where μ_{a_p}, $p = 1, 2, 3$, are the Beltrami differentials dual to the moduli $(d\tau = d\Omega_{11} \, , d\bar{\tau} = d\Omega_{22} \, , dl = d\Omega_{12})$, respectively. The correlation function on the r.h.s.,

$$\langle b(w_1) b(w_2) b(w_3) \rangle \equiv Z_2(w_1, w_2, w_3) = Z_2 \det \omega_i \omega_j (w_p) \, , \tag{B.2}$$

where the constant factor Z_2 can be considered as the (oscillator part of the) partition function of the (2,−1) $b{-}c$ system. This determinant has already been included in the amplitude (4.6), therefore in order to avoid double counting we inserted the factor Z_2^{-1} in (B.1). The three quadratic differentials, $\omega_1\omega_1$, $\omega_1\omega_2$ and $\omega_2\omega_2$ form a basis of the b-ghost zero modes. After inserting (B.2) into the r.h.s. of (B.1), the position integration can be performed explicitly; since the Beltrami differentials are dual to $d\Omega_{ij}$,

$$d\mu(\Omega) = \; d\tau_1 d\tau_2 \, dl \, . \tag{B.3}$$

In order to determine the large τ_2 behavior of Z_2, it is convenient to use the bosonisation formula [15]:

$$Z_2(w_1, w_2, w_3) = Z_1^{-1/2} \theta(\sum_{p=1}^{p=3} w_p - 3\Delta) \prod_{p<q}^{1,2,3} E(w_p, w_q) \prod_{p=1}^{p=3} \sigma^3(w_p) \, , \tag{B.4}$$

where the factor $Z_1^{-1/2}$ is the (oscillator part of the) partition function of a chiral scalar field. The argument of the theta function can be rewritten as:

$$\sum_{p=1}^{p=3} w_p - 3\Delta = \vec{\xi} + 3\delta \; , \qquad\qquad \vec{\xi} = \sum_{p=1}^{p=3} w_p - 3\Delta_\delta \; . \tag{B.5}$$

Since $\vec{\xi}$ is finite in the $\tau_2 \to \infty$ limit, the leading term of the theta series is

$$\theta(\sum_{p=1}^{p=3} w_p - 3\Delta) = e^{4\pi(\tau_2 - l)} f(\vec{\xi}) \, , \tag{B.6}$$

where

$$f(\vec{\xi}) = e^{-2\pi(\xi_1+\xi_2)}[1 + e^{-\pi l}e^{-2\pi i\xi_1} + e^{-\pi l}e^{-2\pi i\xi_2} + e^{-2\pi i(\xi_1+\xi_2)}]\,. \tag{B.7}$$

It is a matter of straightforward but quite tedious algebra to show that for the differentials (5.1),

$$\det \omega_i\omega_j(w_p) = f(\vec{\xi})\ (e^{2\pi l} - 1)^2\, e^{-6\pi l} \prod_{p<q}^{1,2,3} E(w_p, w_q) \prod_{p=1}^{p=3} \sigma^3(w_p)\,. \tag{B.8}$$

Hence, after comparing Eqs.(B.2) and (B.4), we obtain

$$Z_2 = Z_1^{-1/2}\, \frac{e^{4\pi(\tau_2-l)}e^{6\pi l}}{(e^{2\pi l}-1)^2}\,. \tag{B.9}$$

At this point, it remains to determine the large τ_2 limit of the chiral boson partition function Z_1. To that end, it is convenient to use the chiral bosonisation formula [15]:

$$Z_1^{3/2} = \frac{\theta(u_1 + u_2 - v - \Delta)}{\det \omega_i(u_j)} \frac{E(u_1, u_2)}{E(u_1, v)E(u_2, v)} \frac{\sigma(u_1)\sigma(u_2)}{\sigma(v)}\,, \tag{B.10}$$

where u_1, u_2 and v are arbitrary. After taking the limit in a similar way as before, we obtain

$$Z_1 = 1\,. \tag{B.11}$$

In order to compare with the existing literature, we note that, according to our results, the "two-loop cosmological constant" of $D = 26$ bosonic string,

$$\frac{Z_2}{Z_1^{13}}\, d\mu(\Omega) = d\tau_1 d\tau_2\, dl\, \frac{e^{4\pi(\tau_2-l)}e^{6\pi l}}{(e^{2\pi l}-1)^2}\,, \tag{B.12}$$

where we used Eqs.(B.3), (B.9) and (B.11). This quantity should be compared with the formal square root of the well-known result [24,25]:

$$\frac{Z_2}{Z_1^{13}}\, d\mu(\Omega) = \prod_{i\le j} \frac{d\Omega_{ij}}{|\Psi_{10}(\Omega)|}\,, \tag{B.13}$$

where Ψ_{10} is the weight 10 generator of the Isgusa ring, $\Psi_{10}(\Omega) = \prod_\xi \theta^2[\xi](\Omega, 0)$, with the product including all 10 even spin structures. Indeed, Eq.(B.12) follows in the $\tau_2 \to \infty$ (l fixed) degeneration limit of Ψ_{10}. The double pole appearing in the factorization limit $\Omega_{12} = l = 0$ can be attributed to the tachyon [25].

Appendix C. Twisted Determinant $\widetilde{\mathbf{Z}}_2$

In order to find the $\tau_2 \to \infty$ limit of the determinant $\widetilde{Z}_2$, we will compare the degeneration limits of the left- and right-hand sides of Eq.(4.26).

On the l.h.s., we have a theta function series, with the leading term

$$\theta\left[\begin{matrix}0\\ \frac{1}{2}\vec{1}\end{matrix}\right](\sum_{a=1}^{a=3} z_a - 3\Delta) = e^{4\pi(\tau_2 - l)}\tilde{f}(\vec{\zeta})\ , \qquad \vec{\zeta} = \sum_{a=1}^{a=3} z_a - 3\Delta_\delta\ , \tag{C.1}$$

where

$$\tilde{f}(\vec{\zeta}) = e^{-2\pi(\zeta_1+\zeta_2)}[1 - e^{-\pi l}e^{-2\pi i\zeta_1} - e^{-\pi l}e^{-2\pi i\zeta_2} + e^{-2\pi i(\zeta_1+\zeta_2)}]\,. \tag{C.2}$$

Furthermore, the degeneration limits of the prime form and the σ-differential are given by Eqs.(5.11) and (5.13), respectively, and $Z_1 = 1$, as shown in Appendix B.

On the r.h.s., we have the determinant $\det \widetilde{h}_a(z_b)$ involving twisted two-differentials $\widetilde{h}_a$, $a = 1, 2, 3$. We can compute this determinant explicitly by choosing the basis

$$\widetilde{h}_1 = \omega_N\,\widetilde{\omega} \qquad \widetilde{h}_2 = \omega_h\,\widetilde{\omega}_{h^{-1}} \qquad \widetilde{h}_3 = \omega_{h^{-1}}\widetilde{\omega}_h\ , \tag{C.3}$$

with the ω-differentials listed in Eqs.(A.4), (A.6), (A.7) and (5.1). After a straightforward but tedious computation, we find that, up to a numerical factor,

$$\det \widetilde{h}_a(z_b) = \mathrm{Im}(h) \times\ \tilde{f}(\vec{\zeta})\,(e^{2\pi l} - 1)\,e^{-6\pi l}\prod_{a<b}^{1,2,3} E(z_a, z_b)\prod_{a=1}^{a=3}\sigma^3(z_a)\,. \tag{C.4}$$

By comparing the two sides of Eq.(4.26), we find:

$$\widetilde{Z}_2 = \frac{e^{4\pi(\tau_2 - l)}\,e^{6\pi l}}{e^{2\pi l} - 1}\,. \tag{C.5}$$

Then Eq.(5.15) follows after combining the above result with Eqs.(B.9) and (B.11). The $\mathrm{Im}(h)$ factor, which reflects antisymmetry of the determinant under $h \leftrightarrow h^{-1}$ for a specific choice of basis (C.3), is irrelevant to the present derivation, however it can be important when summing over twisted sectors, as pointed out in Section 6.

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